Backward λ-lemma and Morse filtrations

Proceedings
IX Nonlinear differential equations
on the occasion of the 60th birthday of Bernhard Ruf

17-21 September 2012
João Pessoa, Paraíba, Brazil

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Abstract

Consider the infinite dimensional hyperbolic dynamical system provided by the (forward) heat semiflow on the loop space of a closed Riemannian manifold M. We use the recently discovered backward λ-lemma and elements of Conley theory to construct a Morse filtration of the loop space whose cellular filtration complex represents the Morse complex associated to the downward L^2-gradient of the classical action functional. This paper is a survey. Proofs and more details will be given in [6].

Contents

1 Introduction

Consider a closed smooth manifold M of dimension $n \geq 1$ equipped with a Riemannian metric M and the Levi-Civita connection ∇. Pick a smooth function $V : S^1 \times M$ and set $V_t(q) := V(t, q)$. Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$.

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For smooth maps $\mathbb{R} \times S^1 \to M : (s,t) \mapsto u(s,t)$ consider the *heat equation*

$$\partial_s u - \nabla_t \partial_t u - \nabla V_t(u) = 0. \tag{1}$$

It corresponds to the downward L^2-gradient equation of the *action* given by

$$\mathcal{S}_V(x) = \int_0^1 \left(\frac{1}{2} |\dot{x}(t)|^2 - V(t, x(t)) \right) dt$$

for any element $x : S^1 \to M$ of the *free loop space* $\Lambda M := W^{1,2}(S^1, M)$ consisting of absolutely continuous loops in M. Consider the solutions $x \in \Lambda M$ of the ODE $-\nabla_t \dot{x} - \nabla V_t(x) = 0$, that is the set of (perturbed) closed geodesics. These are the critical points of $\mathcal{S}_V$, because $-\nabla_t \dot{x} - \nabla V_t(x) = \operatorname{grad} \mathcal{S}_V(x)$ where grad denotes the L^2-gradient. *Throughout this paper* we fix a regular value a of $\mathcal{S}_V$ and assume that the Morse-Smale condition holds true below level a. Consider the sublevel set $\Lambda^a M := \{\mathcal{S}_V < a\}$. In this case the action is a Morse function on $\Lambda^a M$ and the set of solutions to (1) that converge to critical points $x^\pm \in \Lambda^a M$, as $s \to \pm\infty$, carries the structure of a smooth manifold whose dimension is given by the Morse index difference $\operatorname{ind}_V(x) - \operatorname{ind}_V(y)$. Consider the set Crit_k of critical points of $\mathcal{S}_V$ in $\Lambda^a M$ whose Morse index is k. Due to the imposed bound a the set Crit of all critical points in $\Lambda^a M$ is finite. For each $x \in \operatorname{Crit}$ pick an orientation of the largest subspace E_x of the Hilbert space

$$X := T_x \Lambda M = W^{1,2}(S^1, x^* TM)$$

on which the Hessian of $\mathcal{S}_V$ at x is negative definite. (The dimension of E_x is finite and called the *Morse index* of x.)

Heat Flow homology [4]

By definition the *Morse chain groups* $\mathrm{CM}_k = \mathrm{CM}_k(\Lambda^a M, -grad\, \mathcal{S}_V)$ are the free abelian groups generated by the (perturbed) closed geodesics x of Morse index k and below level a, that is $\mathbb{Z}^{\operatorname{Crit}_k}$. Set $CM_k = \{0\}$ in case of the empty set. The chosen orientations provide a so-called *characteristic sign* $n_u \in \{\pm 1\}$ for each heat flow solution u of (1) between critical points of index difference one. Up to shift in the time variable s, there are only finitely many such u. Counting them with signs n_u provides the *Morse boundary operator* $\partial_k : \mathrm{CM}_k \to \mathrm{CM}_{k-1}$. By HM_k we denote the corresponding homology groups.

Main result: The natural isomorphism to singular homology [6]

The idea to use cellular filtrations to calculate Morse homology goes back at least to Milnor [3]. One needs to construct a cellular filtration $\mathcal{F}$ of $\Lambda^a M$ whose cellular filtration complex $(\mathrm{C}_* \mathcal{F}, \partial_*)$ precisely represents the Morse complex, up to natural identification. In this case we are done, namely

$$\mathrm{HM}_k \equiv \mathrm{H}_* \left((\mathrm{C}_* \mathcal{F}, \partial_*) \right) \simeq \mathrm{H}_*(\Lambda^a M) \tag{2}$$

where the identity takes place already on the chain level and the natural isomorphism $\simeq$ is provided by algebraic topology for any cellular filtration; see e.g. [2]. Here and throughout homology is understood with integer coefficients.

2 Morse filtrations and Conley pairs

Definition 2.1 (Cellular filtration and homology)**.** Assume $\mathcal{F} = (F_0 \subset F_1 \subset \dots \subset F_{n_a})$ is a nested sequence of open subsets of $\Lambda^a M$ such that relative singular homology $H_\ell(F_k, F_{k-1})$ is trivial whenever $\ell \neq k$. Set $F_{-1} := \emptyset$. Then $\mathcal{F}$ is (a special case of) a *cellular filtration of* $\Lambda^a M$. For the algebraic topology used in this section we refer to [2]. The *cellular chain complex* consists of the *cellular chain groups* $\mathrm{C}_k\mathcal{F} := \mathrm{H}_k(F_k, F_{k-1})$ together with the triple boundary operator $\partial_k : \mathrm{H}_k(F_k, F_{k-1}) \to \mathrm{H}_{k-1}(F_{k-1}, F_{k-2})$. A cellular filtration $\mathcal{F}$ is called *Morse filtration*, if in addition $H_k(F_k, F_{k-1})$ is the free abelian group CM_k generated by the critical points of Morse index k, that is $\mathrm{C}_k\mathcal{F} = \mathrm{CM}_k$.

Remark 2.2. To prove (2) we need the construct a Morse filtration $\mathcal{F}$ for $\Lambda^a M$ and show that its cellular boundary operator acts by counting heat flow lines with their characteristic signs. Comparison of boundary operators carries over from flows, cf. [3] or [1, thm. 2.11], since our unstable manifolds are of finite dimension and carry a genuine flow. So to prove (2) it remains to construct $\mathcal{F}$.

The Abbondandolo-Majer construction for flows [1]

In their construction of a Morse filtration $\mathcal{F}'$ for $\Lambda^a M$ openness of the sets F'_k follows from openness of the time-T-map and the Morse property is a consequence of forward flow invariance of the open sets F'_k. One begins by setting N_0 equal to the union of *local* sublevel sets of the form $\{\mathcal{S}_V < \mathcal{S}_V(x_0) + \varepsilon\}$ where x_0 runs through all local minima. For $\varepsilon > 0$ sufficiently small N_0 is a disjoint union. Set $F'_0 := N_0$. Now choose a sufficiently small[1] open neighborhood N'_1 of the index one critical points which does not contain any other critical point. Then consider the union of F'_0 and the whole forward flow of N'_1 and call it $F'_1 := F'_0 \cup \varphi_{[0,\infty)} N'_1$. Similarly define F'_2 and $F'_3, \dots F'_{n_a}$.

A new construction for semiflows using Conley pairs [6]

The Cauchy problem associated to the heat equation (1) for maps $[0,\infty) \to \Lambda^a M : s \mapsto u_s = u(s,\cdot)$ is well posed and leads to the continuous *semiflow*

$$\varphi : [0,\infty) \times \Lambda^a M \to \Lambda^a M$$

called the *heat flow.* In fact φ is of class C^1 on $(0,\infty)$. A characteristic feature of the heat flow is its extremely regularizing nature, namely $\varphi_s \gamma$ is C^∞ smooth for each $\gamma \in \Lambda M$ and any time $s > 0$. Observe that the set of nonsmooth elements is dense[2] in ΛM. Hence φ_s is not an open map and the Abbondandolo-Majer method does not work. Instead we propose the following construction.

It is a very simple, but in this case far reaching, observation that *by continuity of φ_s preimages of open sets are open.* As above define N_0 as the union of

[1] *Morse-Smale on neighborhoods*: Roughly speaking, the Morse-Smale property extends to a small neighborhood N of Crit. Pick $N'_1 \subset N$.

[2] Given any $\gamma \in \Lambda M$ consider the sequence $\exp_\gamma(\frac{1}{n}\xi)$ in ΛM where ξ is any small nonsmooth $W^{1,2}$ vector field along γ.

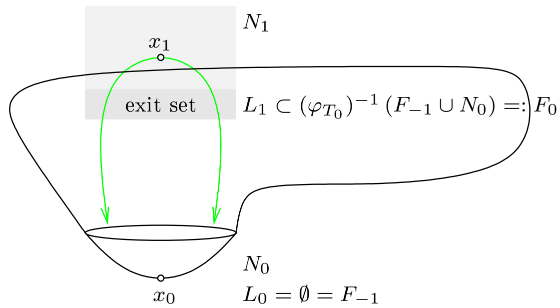

Figure 1: Morse filtration $\mathcal{F} = (\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{n_a} = \Lambda^a M)$

local sublevel sets near the local minima. Then consider an index one critical point x and the preimage $(\varphi_T)^{-1} N_0$ where $T \geq 0$. This preimage is open and semiflow invariant. By Morse-Smale the one-dimensional unstable manifold of x eventually enters[3] N_0. Consequently our preimage gets very close to x whenever T is very large, but for finite T it will never contain x. To get over the obstruction x assume we had an open neighborhood N_x of x containing no other critical points and an open subset $L_x \subset N_x$ whose closure does not contain x. Assume further that L_x is semiflow invariant in N_x and every element leaving N_x under the semiflow necessarily runs through L_x. The pair (N_x, L_x) is called a *Conley pair* and L_x is an *exit set* for the *Conley set* N_x.

Pick $x \in \mathrm{Crit}$ and set $c := \mathcal{S}_V(x)$. For $\varepsilon > 0$ small and $\tau > 0$ large the sets

$$\begin{aligned} N_x = N_x^{\varepsilon,\tau} &:= \left\{\gamma \in \Lambda^{c+\varepsilon} M \mid \mathcal{S}_V(\varphi_\tau \gamma) > c - \varepsilon\right\}_x \\ L_x = L_x^{\varepsilon,\tau} &:= \{\gamma \in N_x \mid \mathcal{S}_V(\varphi_{2\tau}\gamma) < c - \varepsilon\} \end{aligned} \tag{3}$$

are a Conley pair for x and by theorem 3.2 (d) the sets N_x corresponding to different critical points x are disjoint. In (3) we denote by $\{\ldots\}_x$ the path connected component containing x. Set $N_k := \cup_{x \in \mathrm{Crit}_k} N_x$ and similarly for L_k. Now set $F_0 := (\varphi_{T_1})^{-1} N_0 \supset L_1$ where the constant T_1 is sufficiently large such that the inclusion holds true; see figure 1. Set

$$F_k := (\varphi_{T_k})^{-1}(F_{k-1} \cup N_k) \supset L_{k+1}, \qquad k = 0, \ldots, n_a - 1, \tag{4}$$

where each constant T_k is chosen sufficiently large[4] such that the inclusion holds true. Because there are no critical points in the complement of $F_{n_a-1} \cup N_{n_a}$ in $\Lambda^a M$, there is a constant T_{n_a} such that $F_{n_a} := (\varphi_{T_{n_a}})^{-1}(N_{n_a} \cup F_{n_a-1}) = \Lambda^a M$. Observe that each set F_k is open, because N_k and F_{k-1} are. Furthermore, although N_k is *not* semiflow invariant the union $N_k \cup F_{k-1}$ *is*, because the exit set L_k of N_k is contained in F_{k-1}. Openness and semiflow invariance heavily enter calculation (5) in the proof of the Morse filtration property.

[3] By Palais-Smale and Morse for each $\gamma \in \Lambda^a M$ the limit $\gamma_\infty := \lim_{s\to\infty} \varphi_s \gamma$ exists and lies in Crit. If $\gamma \in W^u(x)$, then by Morse-Smale either $\gamma_\infty \in \mathrm{Crit}_0$ or $\gamma = x$.

[4] Here Palais-Smale, Morse-Smale on neighborhoods, and $\mathcal{S}_V$ being bounded below enter.

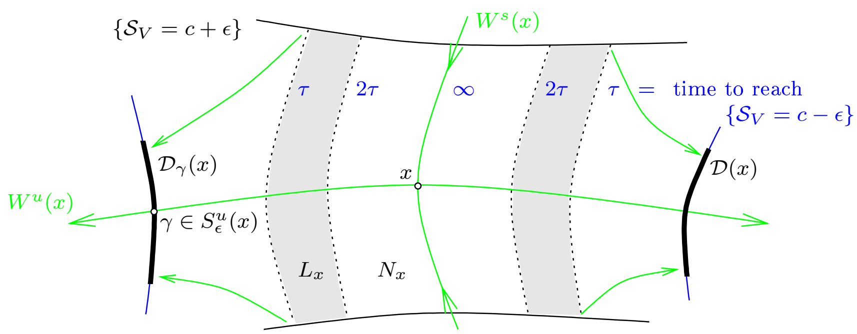

Figure 2: Conley pair (N_x, L_x) foliated by equal time disks $(\varphi_T)^{-1}\mathcal{D}_\gamma(x)$

Morse filtration property

Constructing suitable homotopy equivalences and applying the excision axiom of relative homology one shows that

$$\mathrm{H}_\ell(F_k, F_{k-1}) \simeq \mathrm{H}_\ell(N_k, L_k) \simeq \bigoplus_{x\in \mathrm{Crit}_k} \mathrm{H}_\ell(N_x, L_x). \tag{5}$$

Here the final step uses that N_k is a union of pairwise disjoint sets N_x. So in order to prove that the nested sequence $\mathcal{F}$ consisting of the open semiflow invariant sets F_k defined by (4) is a Morse filtration for $\Lambda^a M$ – thereby concluding the proof of (2) via remark 2.2 – it remains to show that

$$\mathrm{H}_\ell(N_x, L_x) \simeq \mathrm{H}_\ell(\mathrm{D}^k, \partial\mathrm{D}^k) \simeq \begin{cases} \mathbb{Z} & ,\ell = k, \\ 0 & , \text{otherwise}, \end{cases} \tag{6}$$

for every $x \in \mathrm{Crit}_k$. To prove the first isomorphism was precisely the problem which inspired us to come up with the backward λ-lemma in [5]: Since the part of N_x in the unstable manifold $W^u(x)$ is a k-disk and the corresponding part of L_x is homotopy equivalent to the disk boundary, it remains to deformation retract (N_x, L_x) to its part in $W^u(x)$. A very simple, but crucial, observation is that the semiflow φ_s deforms the *ascending disk* $W^s_\varepsilon(x) := W^s(x) \cap \Lambda^{c+\varepsilon} M = W^s(x) \cap N_x$ to x, as $s \to \infty$. Clearly this fails on other parts of N_x. Note that $W^s_\varepsilon(x)$ is a C^1 graph over its tangent space denoted by, say X^+. The idea is to *foliate all of N_x by copies of $W^s_\varepsilon(x)$, more precisely C^1 graphs over X^+, and then extend φ_s artificially to all of N_x using the graph maps*; see (8) and figure 4.

To see the foliation assign to each point of N_x the time T at which it hits the level surface $\{\mathcal{S}_V = c - \varepsilon\}$; see figure 2. This suggests that N_x is foliated by (pieces of) the equal time hypersurfaces $(\varphi_T)^{-1}\{\mathcal{S}_V = c-\varepsilon\}$ for $T \in (\tau, \infty)$. For $T = \infty$ one obtains the codimension $k = \mathrm{ind}_V(x)$ ascending disk $W^s_\varepsilon(x)$. Of course, the leaves of a foliation need to be of the same codimension: Consider the tubular neighborhood $\mathcal{D}(x) \to S^u_\varepsilon(x)$ associated to the (sufficiently small) radius r normal bundle of the descending sphere $S^u_\varepsilon(x) := W^u(x) \cap \{\mathcal{S}_V = c-\varepsilon\}$ in the Hilbert manifold $\{\mathcal{S}_V = c-\varepsilon\}$. Each fiber $\mathcal{D}_\gamma(x)$ is a codimension k disk.

3 Backward λ-lemma and stable foliation

Fix $x \in \mathrm{Crit}_k$ and set $c := \mathcal{S}_V(x)$. Since $N_x = N_x^{\varepsilon,\tau}$ fits into any neighborhood of x for $\varepsilon > 0$ small and $\tau > 0$ large we use local coordinates about $x \in \Lambda M$.

Local coordinates about $x \in \Lambda M$

Observe that paths $s \mapsto u(s)$ in ΛM near x and paths $s \mapsto \xi(s)$ in a closed ball $\mathcal{B}_{\rho_0}$ about $0 \in X = T_x\Lambda M$ uniquely correspond to each other via the identity $u(s) = \exp_x \xi(s)$ pointwise for every $t \in S^1$. In the new coordinates ξ the Cauchy problem associated to (1) turns into the equivalent Cauchy problem

$$\zeta'(s) + A\zeta(s) = f(\zeta(s)), \qquad \zeta(0) = z \in \mathcal{U}, \tag{7}$$

for maps $\zeta : [0, T] \to \mathcal{B}_{\rho_0} \subset X$. Here $A = A_x$ is the Jacobi operator associated to x. The semiflow φ turns into a local semiflow ϕ on $\mathcal{B}_{\rho_0} \subset X$. The nondegenerate critical point x corresponds to the hyperbolic fixed point 0 of ϕ. Furthermore,

$$X = W^{1,2}(S^1, x^*TM) = T_x\Lambda M \simeq T_xW^u(x) \oplus T_xW^s(x) =: X^- \oplus X^+$$

where the splitting is orthogonal and X^- is of finite dimension $\mathrm{ind}_V(x)$ and consists of smooth loops along x. By $\pi_\pm : X \to X^\pm$ we denote the associated (orthogonal) projections. For coordinate representatives of global objects we shall use the global notation omitting x, for example $W^u(x)$ becomes W^u, and we denote $\mathcal{S}_V$ by $\mathcal{S}$. Via a (standard) change of coordinates one achieves that $W^u \subset X^-$ locally near 0. By $\mathcal{B}_r^+$ we denote the radius r ball about $0 \in X^+$. The *spectral gap* $d > 0$ is the distance between 0 and the spectrum of A_x.

Theorem 3.1 (Backward λ-lemma, [5])**.** *Assume the local setup above with ρ_0 being determined by the nonlinear part of (1). Pick $\mu \in (0, d)$ and a hypersurface $\mathcal{D} \subset \mathcal{B}_{\rho_0}$ of the form $S_\varepsilon^u \times \mathcal{B}_r^+$. Then the following is true (see figure 3). There is a ball $\mathcal{B}^+$ about $0 \in X^+$, constants $\mu, T_0 > 0$, and a Lipschitz continuous map*

$$\begin{aligned} \mathcal{G} : (T_0, \infty) \times S_\varepsilon^u \times \mathcal{B}^+ &\to W^u \times \mathcal{B}^+ \subset \mathcal{B}_{\rho_0} \\ (T, \gamma, z_+) &\mapsto \left(G_\gamma^T(z_+), z_+\right) =: \mathcal{G}_\gamma^T(z_+) \end{aligned}$$

of class C^1. Each map $\mathcal{G}_\gamma^T : \mathcal{B}^+ \to X$ is bi-Lipschitz, a diffeomorphism onto its image, and $\mathcal{G}_\gamma^T(0) = \phi_{-T}\gamma =: \gamma_T$. The graph of G_γ^T consists of those $z \in \mathcal{B}_{\rho_0}$ which satisfy $\pi_+ z \in \mathcal{B}^+$ and reach the fiber $\mathcal{D}_\gamma = \{\gamma\} \times \mathcal{B}_r^+$ at time T, that is

$$\mathcal{G}_\gamma^T(\mathcal{B}^+) = (\phi_T)^{-1}\mathcal{D}_\gamma \cap \left(X^- \times \mathcal{B}^+\right).$$

Furthermore, the graph map $\mathcal{G}_\gamma^T$ converges uniformly in C^1, as $T \to \infty$, to the stable manifold graph map $\mathcal{G}^\infty$. More precisely, the estimates

$$\left\|\mathcal{G}_\gamma^T(z_+) - \mathcal{G}^\infty(z_+)\right\|_X \le e^{-T\frac{\mu}{4}}, \qquad \left\|d\mathcal{G}_\gamma^T(z_+)v\right\|_2 \le 2\left\|v\right\|_2,$$

and

$$\left\|d\mathcal{G}_\gamma^T(z_+)v - d\mathcal{G}^\infty(z_+)v\right\|_2 \le e^{-T\frac{\mu}{4}}\left\|v\right\|_2$$

hold true for all $T > T_0$, $\gamma \in S_\varepsilon^u$, $z_+ \in \mathcal{B}^+$, and v in the L^2 closure of X^+.

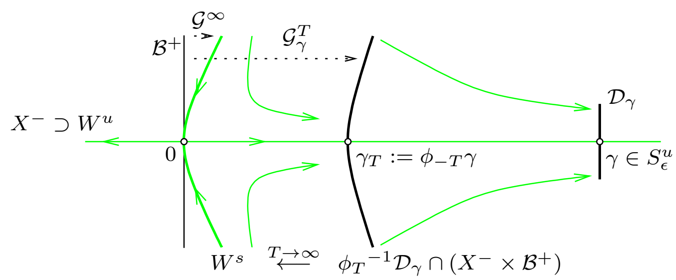

Figure 3: Backward λ-lemma

Theorem 3.1 is based on the observation that the Cauchy problem for a heat flow line $\xi : [0,T] \to X$ with $\xi(0) = z$ is equivalent to a *mixed Cauchy problem* with data (T, γ, z_+). Namely, there is a unique heat flow line $\xi : [0,T] \to X$ with $\pi_+\xi(0) = z_+$ and $\pi_-\xi(T) = \gamma$. But this amounts to *partially solving the heat equation in backward time on certain* open *sets*. Previously backward information could only be obtained on the (k-dimensional) unstable manifolds.

Stable foliation of N_x

Theorem 3.1 foliates (globally meaningless) neighborhoods of x by codimension k disks. The next result provides global information in various directions. By definition the *descending disk* $W^u_\varepsilon(x)$ is given by $W^u(x) \cap \{\mathcal{S}_V > c - \varepsilon\}$.

Theorem 3.2 ([6])**.** *Given $\mu \in (0,d)$ there are constants $\varepsilon_1, \tau_1, r > 0$ such that the following is true. Assume $\tau > \tau_1$ and $\varepsilon \in (0, \varepsilon_1)$ and consider the radius r tubular neighborhood $\mathcal{D}(x) \to S^u_\varepsilon(x)$ defined in the paragraph preceding section 3.*

a) The Conley set $N_x = N_x^{\varepsilon,\tau}$ carries the structure of a codimension k foliation whose leaves are parametrized by the disk $\varphi_{-\tau} W^u_\varepsilon(x)$. The leaf over x is the ascending disk $W^s_\varepsilon(x)$ and the other leaves are given by the disks

$$N_x(\gamma_T) = \left\{(\varphi_T)^{-1}\mathcal{D}_\gamma(x) \cap \{\mathcal{S} < c + \varepsilon\}\right\}_{\gamma_T}, \qquad \gamma_T := \varphi_{-T}\gamma,$$

whenever $T > \tau$ and $\gamma \in S^u_\varepsilon(x)$.

b) Leaves and semiflow are compatible in the sense that

$$z \in N_x(\gamma_T) \quad \Rightarrow \quad \varphi_\sigma z \in N_x(\varphi_\sigma \gamma_T)$$

whenever $\sigma \in [0, T - \tau)$.

c) The leaves converge uniformly to the ascending disk in the sense that

$$\operatorname{dist}_{W^{1,2}}\left(N_x(\gamma_T), W^s_\varepsilon(x)\right) \le e^{-T\frac{\mu}{4}}$$

for all $T > \tau$ and $\gamma \in S^u_\varepsilon(x)$. Furthermore, if U is a δ-neighborhood of $W^s_\varepsilon(x)$ in ΛM, then $N_x^{\varepsilon,\tau_*} \subset U$ for some constant τ_*.

d) Assume U is an open neighborhood of x in ΛM. Then there are constants ε_* and τ_* such that $N_x^{\varepsilon_*,\tau_*} \subset U$.

4 Strong deformation retracts

Pick $x \in \mathrm{Crit}_k$. It remains to prove (6). If $k = 0$, then $L_x = \emptyset$ and $\{x\} = W^u(x)$ is a strong deformation retract of $W^s_\varepsilon(x) = N_x$. The retraction is provided by the semiflow φ_s itself and we are done. Now assume $k > 0$. Consider the local setup of section 3 in which N_x is denoted by N and similarly for other quantities. Pick $\rho_0 > 0$ small enough such that the only critical point in $\mathcal{B}_{\rho_0}$ is 0.

Definition 4.1. By theorem 3.2 each $z \in N$ lies on a leaf $N(\gamma_T)$ for some time $T > 0$ and some point γ in the descending disk S^u_ε and where $\gamma_T := \phi_{-T}\gamma$. The continuous leaf preserving map $\theta : [0, \infty) \times N \to N$ defined by

$$\theta_s z := \mathcal{G}^T_\gamma \pi_+ \phi_s \mathcal{G}^\infty \pi_+ z \tag{8}$$

is called the *induced semiflow on* N; see figure 4. It is of class C^1 on $(0, \infty) \times N$.

That θ_s preserves the central leaf $N(0) = W^s_\varepsilon$ is due to the downward L^2-gradient nature of the heat equation. The proof for a general leaf $N(\gamma_T)$ turns out to be surprisingly complex although the idea is once more simple: Show that the map $s \mapsto \mathcal{S}(\theta_s z)$ strictly decreases whenever z lies in the (topological) boundary of a leaf. This implies preservation of leaves as follows. Firstly, note that θ is actually defined on a neighborhood of $N(\gamma_T)$ in $\mathcal{G}^T_\gamma(\mathcal{B}^+)$. Secondly, the (topological) boundary of a leaf lies on action level $c + \varepsilon$ whereas the leaf itself lies strictly below that level. Thus the induced semiflow points inside along the boundary of each leaf – which is a disk by theorem 3.2. So θ_s preserves leaves, thus N and L by theorem 3.2. Moreover, it continuously deforms both topological spaces to their respective part in the unstable manifold and this concludes the proof of (6). Therefore $\mathcal{F}$ defined by (4) is indeed a Morse filtration for $\Lambda^a M$ and by remark 2.2 this establishes the desired natural isomorphism (2).

It remains to show that $\frac{d}{ds}\mathcal{S}(\theta_s z) < 0$ whenever z lies in the (topological) boundary of a leaf. Note that $\mathrm{grad}\mathcal{S}$ is defined on loops whose regularity is at least $W^{2,2}$. Consider the neighborhood $\mathcal{W} := \mathcal{B}_{\rho_0} \cap \{\mathcal{S} \le c + \varepsilon/2\}$ of 0 illustrated

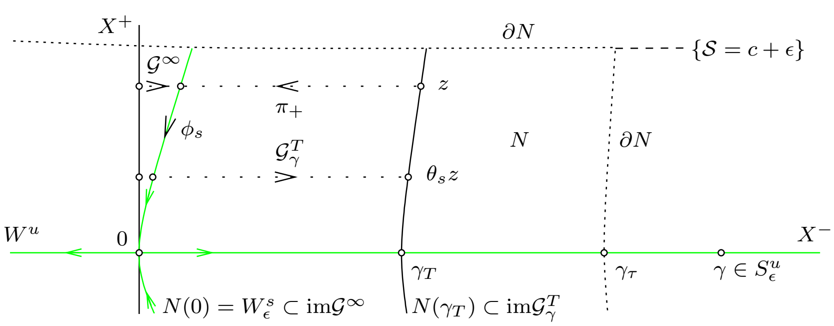

Figure 4: The induced flow θ_s on N

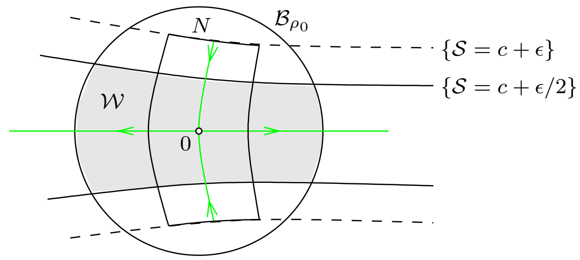

Figure 5: The neighborhood $\mathcal{W}$ of 0 used to define $\alpha > 0$

by figure 5. By Palais-Smale the constant defined by

$$\alpha := \inf_{z\in\left(\mathcal{B}_{\rho_0}\cap W^{2,2}\right)\setminus\mathcal{W}} \left\|\mathrm{grad}\mathcal{S}(z)\right\|_2 > 0$$

is strictly positive. A rather technical argument, see [6], involving a long calculation which uses heavily the estimates provided by theorem 3.1 shows that for all $\varepsilon > 0$ small and $\tau > 0$ large the following is true. If $T > \tau$ and $\gamma \in S^u_\varepsilon$, then

$$\begin{aligned}\frac{d}{ds}\mathcal{S}(\theta_s z) &= d\mathcal{S}|_{\theta_s z}\, d\mathcal{G}^T_\gamma|_{z_+(s)}\, \pi_+ \tfrac{d}{ds}\left(\phi_s \mathcal{G}^\infty \pi_+ z\right)\\ &= -\left\langle \mathrm{grad}\mathcal{S}|_{\theta_s z}, d\mathcal{G}^T_\gamma|_{z_+(s)}\pi_+ \mathrm{grad}\mathcal{S}|_{\phi_s q}\right\rangle_{L^2}\\ &\le -4\alpha^2\end{aligned}$$

for all $z \in \partial N(\gamma_T)$ and $s > 0$ small. It is precisely this calculation where we need the extension to L^2 of the linearized graph map $d\mathcal{G}^T_\gamma(z_+)$ in theorem 3.1.

Acknowledgements. The author would like to thank Alberto Abbondandolo, Kai Cieliebak, and Klaus Mohnke for most useful discussions and comments.

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